ADVANCED FINITE DIFFERENCE METHOD (FDM) FOR COMPUTATIONAL FINANCE

DATASIM
EDUCATION BV

Day 1 Foundations: PDEs in Computational Finance

In this part we lay the foundations for the Finite Difference Method (FDM) by introducing the precise mathematical specifications for several classes of Partial Differential Equations (PDE) and in particular those PDEs that we encounter in computational finance. We analyse the PDEs in order to specify them unambiguously (and without *handwaving*) and to allow a smooth transition to the FDM phase. We also discuss how to transform or *preprocess* the PDE in different ways in order to resolve certain numerical difficulties. We then discuss these PDEs in the light of computational finance.

Boundary Value Problems for Laplace's Equation

- Rectangular, circular and spherical regions
- Rectangular, polar, cylindrical and spherical coordinates
- Spherical harmonics
- Numerical solution of Laplace's equation

First-Order Hyperbolic PDEs

- Categories
- Number of independent variables
- Scalar equations; systems of equations
- Linear, semilinear and quasilinear equations
- Initial value problems
- Nonexistence and nonuniqueness of solutions

Categories of Parabolic PDEs

- Diffusion Equations
- Diffusion-reaction
- Convection-diffusion
- Convection-diffusion-reaction
- Examples and applications

Initial Boundary Value Problems (IBVP)

- Defining the space domain
- Cauchy problem on infinite and semi-infinite domains
- Bounded domain
- Domain truncation/domain transformation
- Boundary conditions (Dirichlet, Neumann, Robin, non-local)

Qualitative Properties of Parabolic PDEs

- Maximum principle
- Uniqueness of solution of IBVPs
- Estimates on solution growth
- Energy inequalities

An Introduction to Free Boundary Problems

- Elliptic variational inequalities
- Existence and regularity
- Obstacle problem
- Filtration problem
- Applications in computational finance

An Introduction to Moving Boundary Problems

- Parabolic variational inequalities
- Introductory examples
- The Stefan problem
- Applications in computational finance

Day 2 Fundamental Finite Difference Method (FDM)

We introduce FDM by analysing it from a number of mathematical and numerical viewpoints. The approach uses state-of-the-art methods to prove stability and accuracy of finite difference schemes, including previous original research and other investigations by the trainer. We focus on one-factor PDEs by discretising them using a range of finite difference schemes. In this way we gain insights into how and why FDM works and how to generalise and incorporate the schemes to n-factor problems. We discuss both linear and nonlinear problems.

FDM: Major Use Cases

- Replacing derivatives by divided differences
- Treatment of initial and boundary conditions
- Stability and convergence
- Assembling and solving the system of equations

Early Exercise Features

- Free and moving boundaries
- Formulations (fixed domain, front tracking)
- Variational inequalities and PSOR
- Brennan-Schwartz constraint checker
- Penalty methods

Some Well-known Schemes
- Explicit and implicit Euler
- Crank Nicolson
- Richardson extrapolation
- Alternating Direction Explicit (ADE)
- Monotone schemes and M-matrices

ADE for one-Factor Problems
- Background and motivation
- Saul'yev, Barakat-Clark and Larkin variants
- ADE for convection terms
- Conditional consistency; unconditional stability
- Boundary conditions

Kinds of Boundary Conditions
- Dirichlet, Neumann, Robin
- Linearity
- PDE on boundary (hyperbolic, parabolic)
- Fichera conditions

Day 3 Advanced Finite Difference Method (FDM) in Finance
In this we focus on two-factor (space) time-dependent PDEs and their approximation by popular finite difference schemes, in particular, *Alternating Direction Implicit* (ADI), *Soviet Splitting* (Locally One Dimensional (LOD)), *Method of Lines* (MOL) and *Alternating Direction Explicit* (ADE) methods. We thus have several proven methods that can be used to solve the problem at hand. We discuss both linear and nonlinear problems and we propose suitable finite difference schemes for both categories.

The Method of Lines (MOL) Overview
- Semi-discretisation
- Vertical MOL and horizontal MOL (Rothe's method)
- Example: one-dimensional heat equation
- Advantages of MOL
- Application areas in computational finance

MOL in Detail
- Stiff and non-stiff ODEs
- Linear and nonlinear systems
- Incorporating non-Dirichlet boundary conditions into MOL
- Adaptive and non-adaptive ODE solvers

MOL PDE Examples
- Black Scholes
- Cox Ingersoll Ross (CIR)
- Uncertain Volatility Model (UVM)
- CEV model
- Pde for credit value adjustment (CVA)
- MOL in Mathematica and Boost C++ *odeint*

The ADI Method
- Using ADI for two-factor PDE
- Mixed derivatives using Craig-Sneyd
- Test cases: basket options and Heston model
- Generalising the ADI method

The Operator Splitting Method
- Yanenko, Marchuk and Strang splittings
- Explicit and implicit splitting
- Handling mixed derivatives and boundary conditions
- Splitting and predictor-corrector methods
- Marchuk 1-2-2-1 model

The ADE Method
- Origins and background; how it differs from ADI and splitting
- Motivating ADE: from heat pde to convection-diffusion and mixed derivatives
- One-sided and centred variants of ADE
- ADE in 3 factors

Comparing ADI, Splitting and ADE Methods
- How they handle mixed derivatives
- Boundary conditions
- Accuracy and robustness of the schemes
- Improving accuracy
- Can the scheme be parallelised?

Mixed Derivatives
- Modeling correlation: extreme cases
- Craig-Sneyd, Verwer, Hout_Welfert, Yanenko
- Stress-testing mixed derivatives
- Test case: compare ADI, splitting and ADE for Heston model

Day 4 Applications and FDM Project A-Z
In this part we apply the results of the first three days to analyse, solve and test PDE/FDM models in computational finance. We examine several problems in computational finance, choosing PDEs to model them and then determining which finite difference schemes to use based on requirements such as efficiency, accuracy and applicability to a range of problems, for example. Some possible projects that the student can choose can vary from equities to fixed income and hybrid models. We also provide C++ code that can be customised for the current project.

Test Cases
- Classic one-factor Black Scholes PDE
- Two-factor basket options
- Cox-Ingersoll-Ross (CIR)
- Heston and SABR model
- Convertible bonds
- Asian options

Computing Sensitivities
- Forward, backward and centred approximations
- Truncation and roundoff errors
- Scalar and vector cases
- Approximating the gradient
- Catastophic cancellation

The Complex Step Method
- Semiautomatic differentiation
- Motivation: scalar functions for first derivative
- Resolving subtractive (catastrophic) cancellation
- Approximating second derivative
- Extending the method to compute gradient and Hessian
- Directional derivatives and Jacobian
- Computing Fréchet derivatives

Applying the Complex Step Method
- Computing sensitivities in finance, engineering and science
- First and second-order option greeks
- Combining the method with analytical solution
- Choice: Complex Method versus Automatic Differentiation
- Engineering applications

Automatic Differentiation (AD)
- Introduction and motivation
- Forward mode
- Reverse mode
- Using the *DiffSharp* package in C# and F#

Basic Accuracy Testing (Proof-of-Concept)
- Estimating the local truncation error
- The effects of domain truncation and domain transformation
- Which boundary conditions are optimal?
- Choice of matrix solver
- Randomly generating input parameters

Improving Accuracy and Robustness
- Non-smooth payoffs
- Convection-dominance and exponential fitting
- Computing option sensitivities
- Automatic testing: producing a report
- Using Cubic splines

Summary of Nitty Gritty Problems and their Resolution
- The problems with Crank Nicolson; using Rannacher method
- Solving first-order hyperbolic PDEs
- Mixed derivatives and preserving monotonicity of finite difference schemes
- Spatial attenuation and spurious reflections at the boundaries

Testing Accuracy: Sparring Partners
- Is there an analytic/quasi-analytic solution to test against?
- Monte Carlo solution
- Binomial method solution
- Two-asset, Heston analytic solution

Test Case: Valuation of Convertible Bonds

The goal of this part is to analyse and design convertible bonds (CB) using the numerical methods that we have already discussed in the previous sections of the course. We discuss a range of features (for example, conversion, callability and putability) and how to model them by PDEs and FDM. We also provide code so that you can run it and modify it to support new features.

At the end of this part, we can see the evolution of a solution from analysis through to design and realisation in C++. This is one of the problems running through the course.

Valuation of Convertible and Callable Bonds, PDE Model
- The Tsiveriotis and Fernandes model (coupled PDEs)
- Brennan-Schwartz model (single PDE)
- Convertible bonds with credit risk
- Payoff, boundary conditions and constraints
- Coupon dates and call dates; accrued interest, clean and dirty prices

Valuation of Convertible Bonds, FDM Model
- Choosing between ADE, Crank-Nicolson and MOL
- FDM for single PDE and systems of PDE
- Coupling in the systems of algebraic equations
- PSOR versus penalty method

Valuation of Convertible Bonds, Results and Extensions
- Code parallelisation and design patterns
- Designing flexible constraints
- Comparing the models and schemes
- Convertible bonds with stochastic short rate (two underlying factors)
- Two-factor Hull-White PDE

Your Trainer
Daniel J. Duffy started the company Datasim in 1987 to promote C++ as a new object-oriented language for developing applications in the roles of developer, architect and requirements analyst to help clients design and analyse software systems for Computer Aided Design (CAD), process control and hardware-software systems, logistics, holography (optical technology) and computational finance. He used a combination of top-down functional decomposition and bottom-

up object-oriented programming techniques to create stable and extendible applications.
Previous to Datasim he worked on engineering applications in oil and gas and semiconductor industries using a range of numerical methods (for example, the finite element method (FEM)) on mainframe and mini-computers.
Daniel Duffy has BA (Mod), MSc and PhD degrees in pure and applied mathematics from Dublin University (Trinity College) and has been active in promoting partial differential equation (PDE) and finite difference methods (FDM) for applications in computational finance. He was responsible for the introduction of the Fractional Step (Soviet Splitting) method and the Alternating Direction Explicit (ADE) method in computational finance.
He is also the originator of the exponential fitting method for time-dependent partial differential equations.
He is also the originator of two very popular C++ online courses (both C++98 and C++11/14) on www.quantnet.com in cooperation with Quantnet LLC and Baruch College (CUNY), NYC. He also trains developers and designers around the world. He can be contacted dduffy@datasim.nl for queries, information and course venues, in-company course and course dates